

Visualizing Mathematics - Online -

Konrad Polthier

Technische Universität Berlin
polthier@math.tu-berlin.de

Abstract. Modern mathematical visualization has always been related with special graphics workstation although visualization was always part of mathematics. Here we start from historical roots, bring interactive visualization into the classrooms and create online mathematical publications. The topics include Java applets and online videos, a new electronic journal for geometry models, an interactive mathematical dissertation and online experiments.

1 *Multimedia of the Old Masters*

Over the recent years, mathematical visualization driven by computer graphics has proven to be a successful tool for mathematicians to investigate difficult mathematical problems which seem to be inaccessible by standard mathematical tools.

But visualization is not a new word in mathematics. Mathematical history contains a rich set of visualizations. For example, the famous copper plate engravings of Hermann Amandus Schwarz. The image appeared in his research publication on the Gergonne problem in 1865. He discovered new minimal surfaces which solved long-standing questions in geometry and analysis on the existence of solutions of elliptic boundary value problems. Although his proof was of theoretical nature he found it worthwhile to invest a lot of energy to include visual images of the new surfaces into his research publications. This becomes even more remarkable if his effort is compared to the much less effort one needs today for the generation of computer images. The head of Apollonius shows lines connecting parabolic points on the surface. Felix Klein wanted to find characteristic lines describing the nature of a shape.

One of the most thorough approaches in mathematics to use physical models and experimental instruments in education and research is the famous collection of mathematical models in Göttingen. This model collection already had a long history when Hermann Amandus Schwarz and Felix Klein overtook the direction of the collection. Especially under the direction of Klein the model collection was systematically modernized and completed for the education in geometry and geodesy. This collection was considered so important that Klein exhibited the models on the occasion of

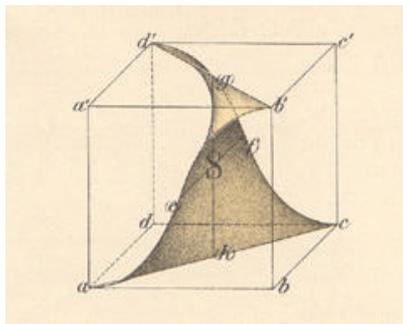


Fig. 1. Left image: Copper plate engraving of the Gergonne surface by Hermann Amandus Schwarz. Image taken from his Collected Works, Springer Verlag 1890. Right image: Head of Apolonius. Image: Hilbert/Cohn-Vossen Anschauliche Geometrie, Springer Verlag.

the World's Columbian Exposition 1893 in Chicago [2]. The models were produced among others by the publisher Martin Schilling in Halle a.S., see his catalog of mathematical models [7]. The price of approximately \$250 per models was relatively high, and therefore the large size of the collection of more than 500 plaster models is even more impressive. The collection can still be seen in the mathematical department in Göttingen, and a description including photos of many models is given in Fischer's book [3].

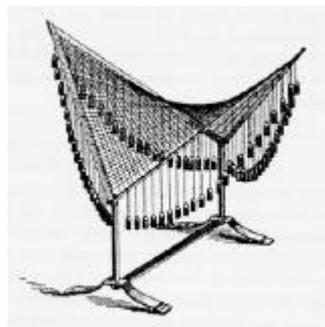


Fig. 2. Left image: Plaster model by Martin Schilling in Halle a.S. Large collection of models collected in Göttingen by Hermann Amandus Schwarz and Felix Klein since 1870. Right image: Steel model of hyperbolic paraboloid from Göttingen Collection. Used in education in geometry and geodesy. Production stopped after 1930. Focus on more abstract ideas in Mathematics, books of van der Waerden and Bourkaki appeared. Images: G. Fischer Mathematical Models, Vieweg Verlag.

2 Gaining Mathematics from Visualization

Among the first breakthroughs of mathematical visualization was the proof of embeddedness of a minimal surface. Celso Costa discovered the mathematical formulae of a genus 1 minimal surface which was a candidate to solve a 200-year long question: whether there exists a third embedded and complete minimal surfaces with finite total curvature beside the trivial examples, the flat plane and the rotational symmetric catenoid. David Hoffman and William Meeks [5] developed computer programs to visualize Costa's surfaces and watch surface properties which they could later successfully prove after having enough insight into the complex shape of the surface.

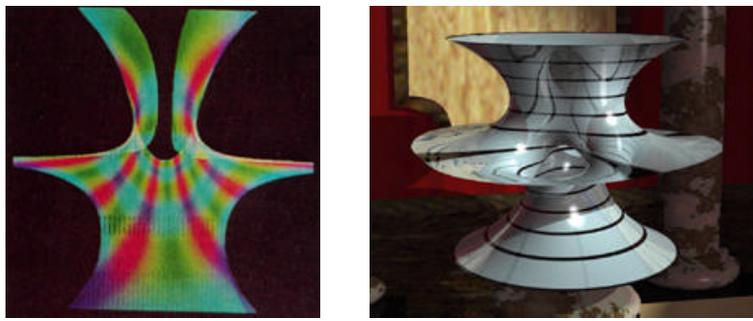


Fig. 3. Costa-Hoffman-Meeks minimal surface (1985). Costa gave formulas of the surface, but how does surface look like, i.e. what geometric properties? For example: does it have self-intersections? The left image shows the very early computer pictures by Hoffman and Meeks.) (left). Years later, we see a raytraced image of the Costa-Hoffman-Meeks minimal surface. (right).

Mathematical visualization has proven to be an efficient tool for analyzing complex mathematical phenomena, and it has given decisive hints leading to rigorous mathematical proofs of long-standing problems. Visualization is not only a tool to visualize complex objects but in combination with modern numerical methods allows to perform mathematical experiments and simulations in an artificially clean environment. For example, the unveiling of the Costa-Hoffman-Meeks surface, or the first numerical examples of compact constant mean curvature surfaces with genus greater than two by Große-Brauckmann and Polthier [4] are among the most prominent results of the fruitful interaction of mathematics with the new toolkit mathematical visualization.

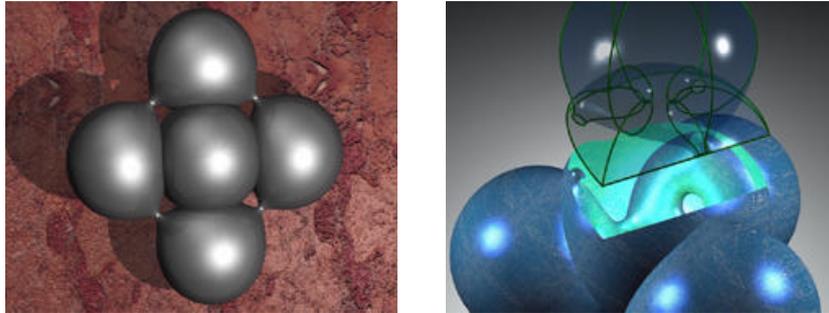


Fig. 4. Compact Constant Mean Curvature Surfaces (1995, GroSse-Brauckmann - P., Oberkapp - P.). Free-boundary value problem for an elliptic partial differential equation. Numerical method based on notion of discrete differential geometry on triangulations. Compact soap bubble with genus four (left) and tetrahedral symmetry (right).

3 Discrete Minimal Surfaces - Online

Minimal surfaces are among the most prominent shapes in mathematics. For an introductory overview and their relation to physical soap films have a look at the online version of a booklet which accompanies the english version of the video *Touching Soap Films*.

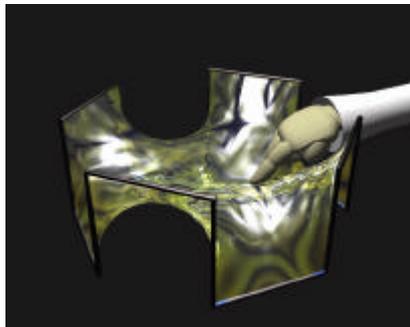


Fig. 5. Snapshot of a minimal soap film taken from the video *Touching Soap Films*.

Discrete surfaces are made up of a set of triangles which are especially convenient when doing numerics and visualization with a computer. In fact, in finite element mathematics and discrete geometry these simplicial complexes have become objects of their own interest. For example, the interactive applets on the online pages of this article work with triangle meshes which are minimal in a discrete sense.

In the following we give several demonstrations of combining interactive experiments, publications and research.

Making Textbooks Interactive A classical text book may easily be enhanced with interactivity. The following page (not shown in the printed book) is taken from the recent introductory textbook on differential geometry "Einführung in die Differentialgeometrie" by Wolfgang Kühnel. Here the static image of a helicoid was replaced with a JavaView applet. Note, the smooth transition from a static book to an interactive book. There is no need for a revolution on the reader's side, for example a changed reading behaviour. The revolution is of technical nature and resides inside the online software hidden to the reader.

Discrete Minimal Catenoid This discrete catenoid is a critical point of the discrete area functional. Discrete minimal surfaces are minimizers of a discrete area functional. The authors (K. Polthier and W. Rossman) have found an explicit representation of discrete catenoids whose set of vertices is dihedrally symmetric. These catenoids are the first explicitly known discrete minimal surfaces for the discrete area functional beside the trivial plane. The explicit representation allows to generate exact discrete minimal surfaces without numerical errors which are especially useful for index computations.

Theorem 1. *There exists a four-parameter family of embedded and complete discrete minimal catenoids $C = C(\theta, \delta, r, z_0)$ with dihedral rotational symmetry and planar meridians. If we assume that the dihedral symmetry axis is the z -axis and that a meridian lies in the xz -plane, then, up to vertical translation, the catenoid is completely described by the following properties:*

1. *The dihedral angle is $\theta = \frac{2\pi}{k}$, $k \in \mathbb{N}$, $k \geq 3$.*
2. *The vertices of the meridian in the xz -plane interpolate the smooth cosh curve*

$$x(z) = r \cosh\left(\frac{1}{r}az\right),$$

with

$$a = \frac{r}{\delta} \operatorname{arccosh}\left(1 + \frac{1}{r^2} \frac{\delta^2}{1 + \cos\theta}\right),$$

where the parameter $r > 0$ is the waist radius of the interpolated cosh curve, and $\delta > 0$ is the constant vertical distance between adjacent vertices of the meridian.

3. *For any given arbitrary initial value $z_0 \in \mathbb{R}$, the profile curve has vertices of the form $(x_j, 0, z_j)$ with*

$$\begin{aligned} z_j &= z_0 + j\delta \\ x_j &= x(z_j) \end{aligned}$$

where $x(z)$ is the meridian in item 2 above.

4. The planar trapezoids of the catenoid may be triangulated independently of each other.

The applet below allows to modify all parameters of the catenoid and study their geometric meaning. Such a web-based applet allows to simultaneously work on this example while one authors is in Germany and the other in Japan.

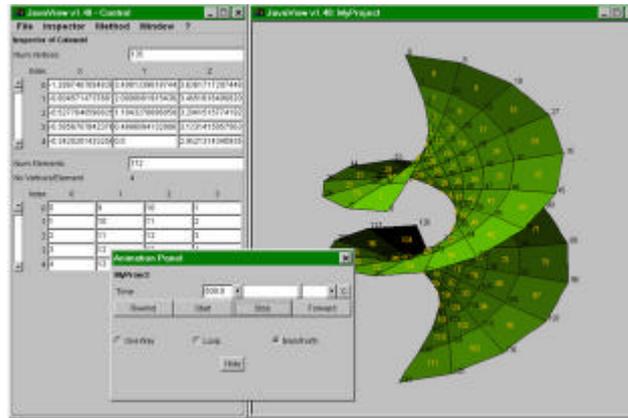


Fig. 6.

Further references on the discrete catenoid may be found on the Electronic Geometry Models server at <http://www.eg-models.de/> model number 2000.05.002.

Discrete Harmonic Maps The research applet on this page minimizes a surface energy. For example, when selecting the Dirichlet energy then the following problem is solved on an arbitrary initial surface:

Find: A minimal surface bounded by a given curve G

Algorithm: Use initial surface M_0 and construct sequence of surfaces M_{i+1} by finding (Laplace-Beltrami) harmonic maps F_i with

$$F_i : M_i \rightarrow M_{i+1} \text{ with } \partial M_{i+1} = G.$$

The limit surface is a minimal surfaces under certain conditions.

During minimization boundary vertices are retained. Pick and drag vertices with the left mouse button while holding key "p" pressed to modify

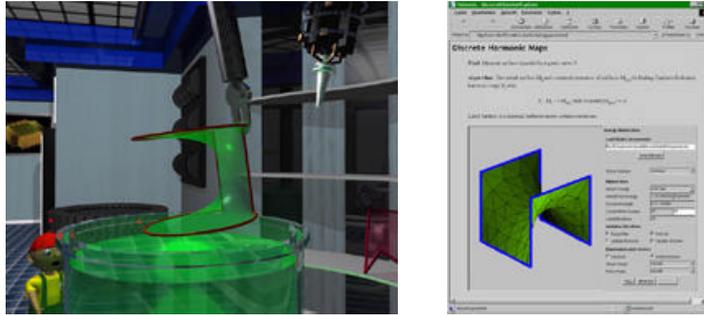


Fig. 7. Soap film machine from video *Touching Soap Films* [1] (left). Computing minimal surfaces in an interactive applet on the JavaView web site [6] (right).

vertices of the surface. Set the number of iteration loops for the minimization algorithm in the textfield "Num Loops".

The checkboxes "Tangential" and "Normal" define in which directions the minimizer is allowed to move vertices. The checkboxes "Update Normals" and "Update Domain" determine if surface normals and domain surface are recomputed in every minimization step. The domain surface in

The button "Step" invokes one minimization step, the button "Minimize" starts as many minimizing iterations as are specified in the "Num Loops" textfield. By the "Resume" button you can stop and continue the iteration.

4 Mathematical Videos

There exists already quite a remarkable set of mathematical videos. The common drawbacks of videos like being non-interactive, requiring special equipment and projection facilities, are nowadays no longer true.

Sequences of videos may be inserted in online documents in the same way as Java applets find their way into mathematical publications. Still, there is a lot of effort required for the production of videos. Have a look at the following sites for online MPEG versions of sequences of the videos.



The VideoMath Festival Cassette is a collection of juried mathematical videos presented at ICM'98, the International Congress of Mathematicians

in Berlin. The videos are winners of a worldwide competition. The mathematical themes include problems in topology and geometry and their recent solutions, visualizations of classical ideas of Archimedes, Eratosthenes, Pythagoras, and Fibonacci, topics in high school mathematics, and applications of modern numerical methods to real world simulations. Length 76 min. (1998). Look at the VideoMath home page for links to many other longer videos.

Touching Soap Films is a popular scientific video on minimal surfaces and soap films. All animations are computer generated. The english edition is distributed by Springer Verlag since 1999. The german edition Palast der Seifenhäute is available through Bild der Wissenschaft and Komplett-Media. Length 41 min. (1995). The Portuguese version was distributed to all schools in Portugal by CMAF/Lisbon in the year 2000.

The video Geodesics and Waves documents recent work on discrete geodesics and their application to numerical problems on curved surfaces and visualization algorithms. It premiered at Siggraph'97 Electronic Theater, received the Science Award of the Multimedia Grand Prix (Nicograph'97) in Japan, and was awarded at other festivals. Length 4:50 min. (1997).

A new mathematical video series was launched by Springer Verlag in June 1999. Among the first films are "Touching Soap Films" by Andreas Arnez, Konrad Polthier, Martin Steffens and Christian Teitzel; "N is A Number - A Portrait of Paul Erdős" by George Paul Csicsery; "Arabesques and Geometry" by Antonio F Costa; and "The New Shepherd's Lamp" by Francois Tisseyre, Claire Weingarten, and Jean-Pierre Bourguignon.

The award-winning computer animations like Outside In, Shape of Space and Not Knot. Videos are available through AK Peters.

Under the direction of Tom Apostol and Jim Blinn Project MATHEMATICS! has produced highly awarded video tapes that explore basic topics in high school mathematics in ways that cannot be done at the chalkboard or in a textbook. The tapes use live action, music, special effects, and imaginative computer animation.

5 EG-Models Journal: An Archive for Electronic Geometry Models

The electronic geometry models archive (at <http://www.eg-models.de>) is a new journal for the publication of digital geometry models. Among the main characteristics are:

- Refereeing of the digital geometry models ensures highest quality of the data sets.

- Each model has a full description and a unique identification number which ensures the intellectual property of the author and allows for a unique citation.
- Possibility to validate geometric experiments by making the data available to 3rd parties through this archive.
- Review of best models in Zentralblatt in a similar way as research articles.

The first collection of reviewed electronic geometry models is open for any geometer to publish new geometric models, or to browse this site for material to be used in education and research. Access to the server is free of charge.

The geometry models in this archive cover a broad range of mathematical topics from geometry, topology, and, to some extent, from numerics. Examples are geometric surfaces, algebraic surfaces, topological knots, simplicial complexes, vector fields, curves on surfaces, convex polytopes, and, in some cases, experimental data from finite element simulations.

All models of this archive are refereed by an international team of editors. The criteria for acceptance follow the basic rules of mathematical journals and are based on the formal correctness of the data set, the technical quality, and the mathematical relevance. This strict reviewing process ensures that users of the EG-Models archive obtain reliable and enduring geometry models. For example, the availability of certified geometry models allows for the validation of numerical experiments by third parties. All models are accompanied by a suitable mathematical description. The most important models will be reviewed by the Zentralblatt für Mathematik.

We are advocating the construction and submission of digital geometric models from various areas of mathematics. The advantages of these digital models go beyond those of the classical plaster shapes and dynamic steel models of earlier days. At the end of the 19th century several mathematicians felt the need to physically handle the geometric objects they thought about. In particular, Felix Klein and Hermann Amandus Schwarz in Göttingen built many models of curves, surfaces and mechanical devices for teaching and other educational purposes.

Using the digital models, interested mathematicians can verify the claims on their own, using appropriate software of their choice. Moreover, once there is a model available, it is possible to perform one's own computational experiments on this data set. This could be a numerical evaluation as well as a search for another property yet to be analyzed for this model.

Each model comes with a detailed description which identifies the author, explains the mathematical purpose, and includes references to other sources of information. Each model has a unique identification number for

unambiguous citation. Each model is equipped with qualified meta data information; therefore, the archive can be searched via specialized search engines such as those from EMIS <http://www.emis.de> and MathNet <http://www.math-net.de/search/germany/>. Each model itself is represented by a master file which is from a fixed set of file formats, including XML formats specified by DTDs. By restricting the data formats we want to ensure that the server's information can be kept up to date on a technical level. Additional files in arbitrary formats are welcome for explanatory purposes.

6 Dissertation Online

Publication of dissertations is a good playground for testing various technical issues related with online documents. JavaView was used to enhance a sample dissertation in mathematics with multimedia elements.

The project Dissertation Online funded by the Deutsche Forschungsgemeinschaft (DFG) is a co-operation of the Humbolt-Universität Berlin, Gerhard-Mercator-University Duisburg, Universität Erlangen-Nürnberg and Carl von Ossietzky-Universität Oldenburg and of the main libraries in Germany. The project defines criteria and guidelines for the digital publishing of dissertations in Germany.

Among the many open problems are:

- Different document structures for different scientific disciplines.
- Designated meta informations for retrieval.
- Copyright and other legal aspects.
- Authentication and unchangeability must be guaranteed.
- Exploit the multimedia capabilities of digital publications and many more issues ...

JavaView has been selected to produce an online version of a mathematical dissertation that demonstrates the possibilities to valorize a thesis with interactive components. The sample dissertation by Tim Hoffmann enhanced with online experiments by Eike Preuß and himself is available at the JavaView homepage.

7 JavaView

JavaView [6] is a 3d geometry viewer and geometry library to publish interactive geometry in any HTML document, and to perform interactive numerical experiments on the internet. A stand-alone version of JavaView also runs as application from a Unix or Dos command prompt, and it can be attached as 3D viewer to other programs.

JavaView's software library focusses on problems in differential geometry and mathematical visualization. Its class library can be used and extended for own numerical experiments in Java, while always profiting from the advanced 3D visualization capabilities and the web integration.

- Tight Integration of Mathematical Experiments, Visualization and Online Publications
- Exchange and Communication of Experiments for research and educational purpose

Since its first public release in November 1999 JavaView was awarded several prizes. As a proof of usability, JavaView was selected by the project "Dissertation Online" of the German science foundation DFG to produce a reference online dissertation in mathematics.

Selected Features of JavaView

- 3D Scientific Viewer in Web Pages
- Software Library for Differential Geometry and Numerics
- Import and Export of Geometries in Multiple Data Formats,
- PostScript PS+EPS; Images GIF, PPM; XML Formats
- Tight integration with Mathematica and Maple.

Images and geometries exported with JavaView may easily be included in TeX publications as well as online documents. Additionally, JavaView is a class library for advanced numerics in differential geometry including tools for finite element numerics. A standalone version of JavaView runs in a Unix or Windows shell from the command prompt, and can be attached as 3D viewer to other programs like *Mathematica*. The JavaView homepage at <http://www.javaview.de> contains several applications and tutorials.

In the following subsections we discuss some properties of JavaView and give a number of sample experiments. Since this book is paper-based we refer to an online version of this article at the web site <http://www-sfb288.math.tu-berlin.de/vgp/> which contains interactive versions of the presented applets.

How to Publish a JavaView Enhanced Web Page? There are different possibilities to add interactive visualizations to a web page using JavaView. Here we assume that a 3d geometry has been produced with any software tool such as Maple, Mathematica, JavaView or others and which was saved in a geometry file `star.mgs`. For an interactive web page we need:

1. The geometry model file `models/star.mgs`
2. Write a web page including the applet below `myPage.html`
3. Get the JavaView library `jars/javaview.jar`
4. Upload all three files to a web server

The sample applet tag somewhere inside the document *myPage.html* looks as follows:

```
<applet
  code=javaview.class
  archive='jars/javaview.jar'
  width=200 height=200>
  <param name=model value='models/stars.mgs'>
</applet>
```

Note, although this example is functional, it is just a demonstration of the required inputs. This applet visualizes the geometry model inside a small window of 200*200 pixels on the web page. The model need not be a geometry file on a local computer but the model parameter may be any internet address referring to a model on an arbitrary web server.

This example stresses the fact that the installation of the JavaView software is no longer an issue compared to the installation process of other software. The browser keeps care to download the required Java archive when it encounters the archive parameter inside the applet tag. The browser also ensures that the archive is downloaded only during first usage, and later reuses the version it has stored in the browser cache.

The easy download mechanism is especially useful for library servers offering Java enhanced electronic publications. The digital article and the JavaView archive are both stored, for example, in the same directory on the library server. The files must be uploaded by the author as described above, and are automatically downloaded by a browser when a user accesses the web page. Therefore, the librarian has no additional duties related with software installation. The library must only offer the usual upload mechanism for documents which it has already installed.

Responsibilities (for Online Publications) Electronic libraries require self-contained (online) publications. Especially, external links must be kept to a minimum in order to allow the library to maintain and ensure the functionality of a document. For example, an interactive dissertation cannot be allowed to access a database outside the library in order to be functional.

Java enabled publications have all required software bundled with the publication. This frees the library from any maintenance task and requires no special support from the provider beside storing the publication which has been uploaded from an author.

Further, on the client side, the reader needs no special software installation to be able to read a Java enhanced publication. The base Java system is installed with custom web browsers and the special modules used in a

publication are downloaded automatically by the web browser when a user accesses the document for the first time.

Author	Create content and upload
Online Library	no special activity, just store the data
Reader	Java enabled browser

For example, the mathematical visualization software JavaView just requires Java 1.1 installation which is automatically available within all modern web browsers.

8 Hardware for 3D Visualization

For a long time scientific visualization was beyond the budget of many mathematical departments. Large research institutes, military organizations, and commercial companies were among the first who could afford specialized graphics hardware. In science, specially funded research groups were able to afford high-end graphics workstations including the necessary staff to manage the machines and simultaneously do the scientific experiment. In the mean-time, the computational power of personal computers with relatively cheap graphics card suffice to perform most of the scientific visualization tasks found in research. Nevertheless, one still encounters the following drawbacks of the current software running on specialized workstations and mainframes:

Workstation Based Specialized and expensive graphics hardware.

- Large program size since operating system just supports basic functionality.
- Usually only the programmer is able to run the experiments.
- Installation at other sites requires experts, and does not allow regular update.
- Advantage: extremely fast execution speed.

These drawbacks are in strong contrast to the situation we have encountered during the development and usage of the software JavaView. JavaView is a scientific visualization software completely written in the programming language Java. Java is an object-oriented programming language similar to the language C and C++ but different in the sense that Java is designed to run on any computer. Further, Java programs may run inside web browsers. Both properties are the reason that Java has become the major programming language for interactive web applications since its first presentation in 1995. A program written in Java has the following advantages:

Online Visualization

- Runs on Standard PC and Workstation.
- Tiny program size because Java base classes are already installed.
- Each application has a user interface per default since it runs in a browser.
- No installation beside a browser with Java since browser performs the data transfer.
- Speed: depends.

These advantages have the following reasons:

(1.) Java is automatically installed on a computer if a web browser is installed. Therefore, the popularity of web browsers helped to install Java on nearly any computer world-wide.

(2.) The size of Java programs is usually very small compared to classical stand-alone application software since the Java base classes, which are comparable to software libraries, are already installed. Therefore, an application must only deliver its additional functionality, and not system routines.

(3.) An application inside a web page must have a well-designed graphical user interface since it is by default used by some other people than the programmer. This is in contrast to classical experimentation software, and leads to a great benefit in the design of better products.

(4.) The installation of classical software systems has often been a pain. The customer often needed to compile the package again on his machine, or make special adjustments depending on his specialized hardware. The author was in an even worse situation. He needed to offer and maintain different versions for different platforms. When using Java then there exists only one version independent of the hardware platform and operating system. This is possible since the Java virtual machine must cope with system differences, so the responsibility is transferred from the author of applications to the supplier of the Java virtual machine. Therefore, the installation process of a Java application such as JavaView is reduced to downloading an archive, i.e. one or more library files, which is done automatically through a web browser. This allows the author to concentrate on the development of the software without keeping to much care on the destination platform, and it frees him from providing installation mechanisms. The user is freed from any installation task, he just starts his browser and selects a Java enhance web page.

(5.) The speed of Java applications not only depends on the hardware but to a large extend on the quality of the installed Java virtual machine (JVM). A Java application consists of machine independent byte code which is interpreted by a JVM and executed on a local computer. JVMs differ

largely in quality, for example, when loading a Java application some JVMs compile the byte code into machine dependent code, which leads to a drastic increase in execution speed.

9 Conclusion

The internet will dramatically change the classical way of communicating and publishing mathematics. We have given some ideas on possible changes to expect, and the benefits which mathematics may gain from these new developments. The interactive, exploratory component of mathematics, which has been removed from mathematical publications for a too long time, is now available in the form of Java enabled software. We have given several examples of multimedia enhanced experiments which allow to imagine the possibilities waiting at the horizon.

For more information and interactive versions of the experiments described in this paper we refer to the JavaView home page. These pages also include tutorial material how to include interactive geometries into own web pages.

A Few Links ...

JavaView	http://www.javaview.de
Mathematical Videos	http://www.springer.de/math/livingmath/
Dissertation Online	http://www-sfb288.math.tu-berlin.de/vgp/dol/
Electronic Geometry Models	http://www.eg-models.de
Mathematics Archives	http://archives.math.utk.edu/
Mathematical Atlas	http://www.math-atlas.org/welcome.html
Scientific Graphics Project	http://www.msri.org/publications/sgp/SGP/

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